Unit 20

Thermal wind, advection, and primitive equations

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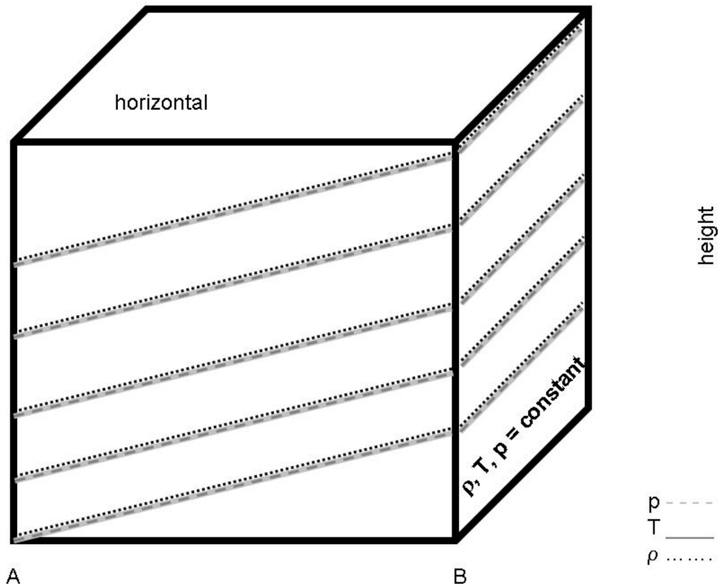
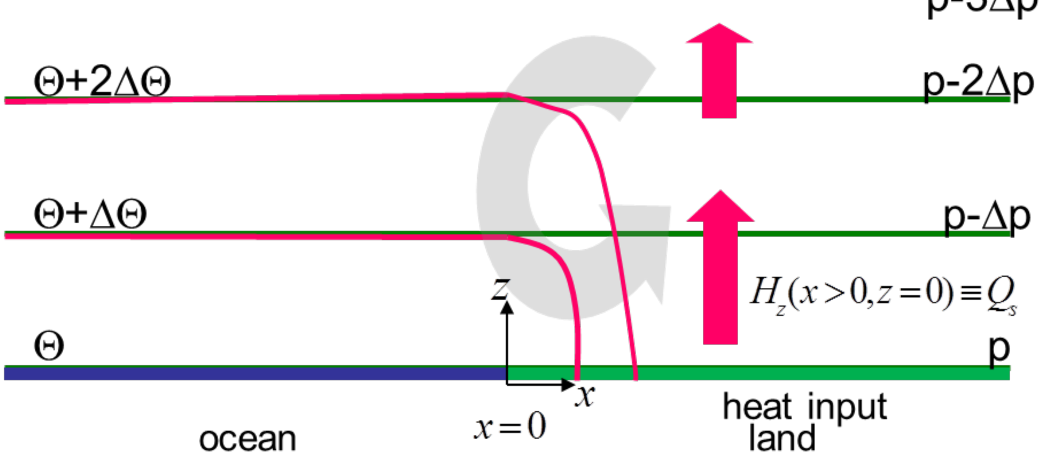
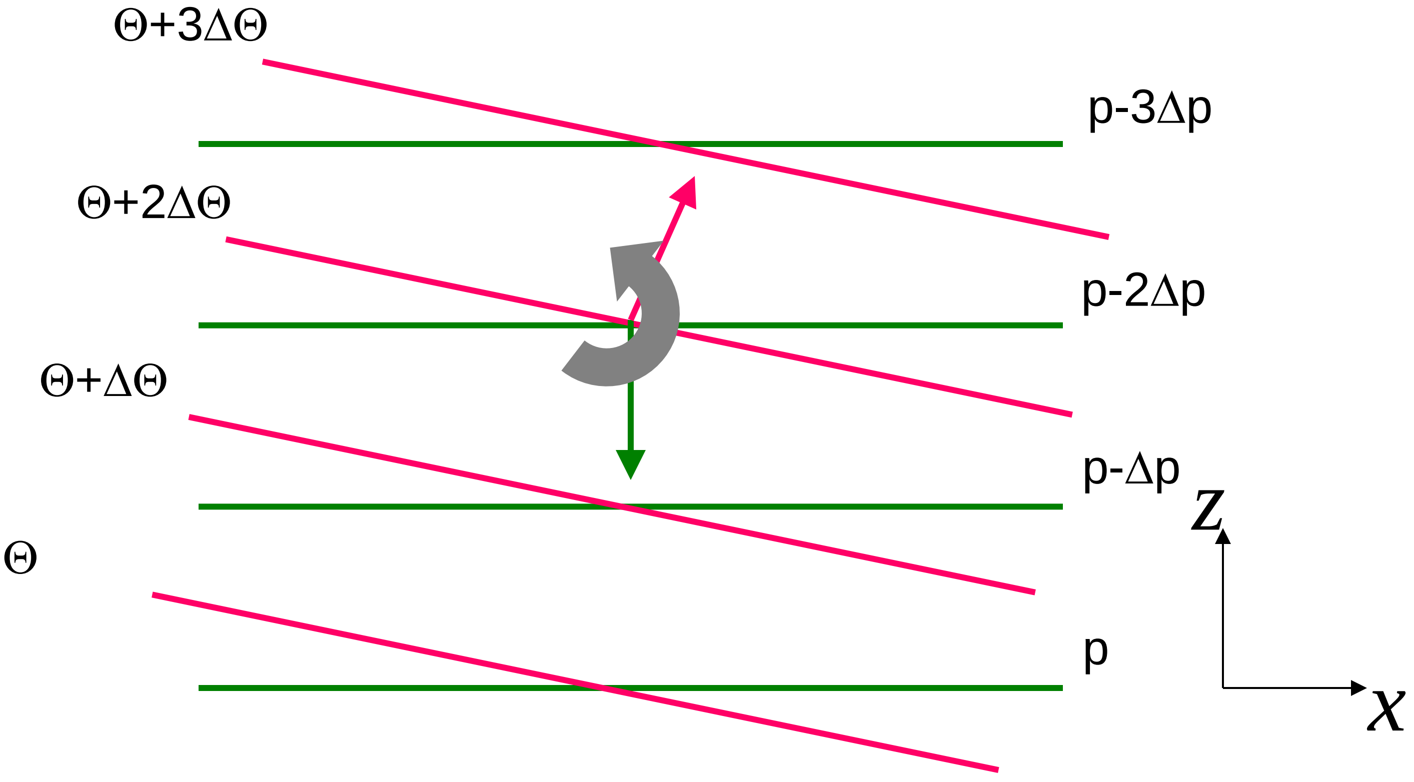
Barotrop vs. barocline





Baroclinicity creates vorticity



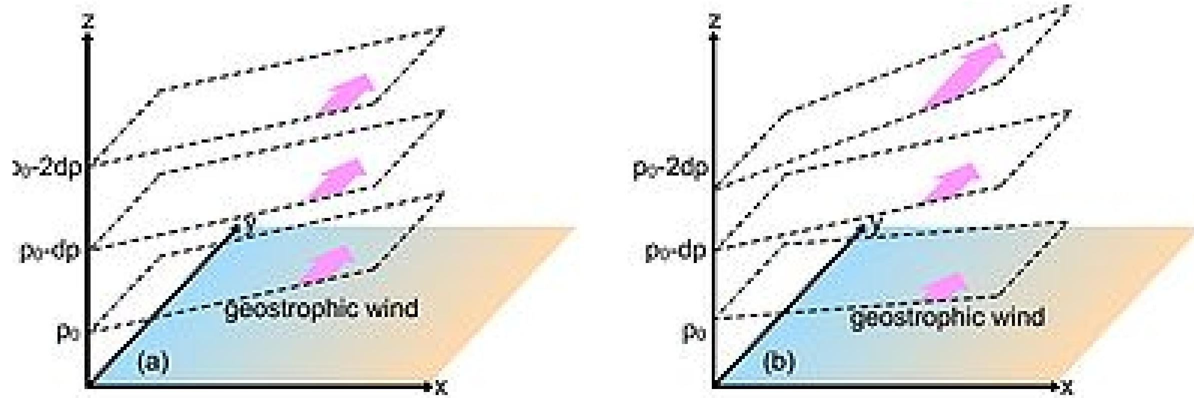
Thermal wind balance

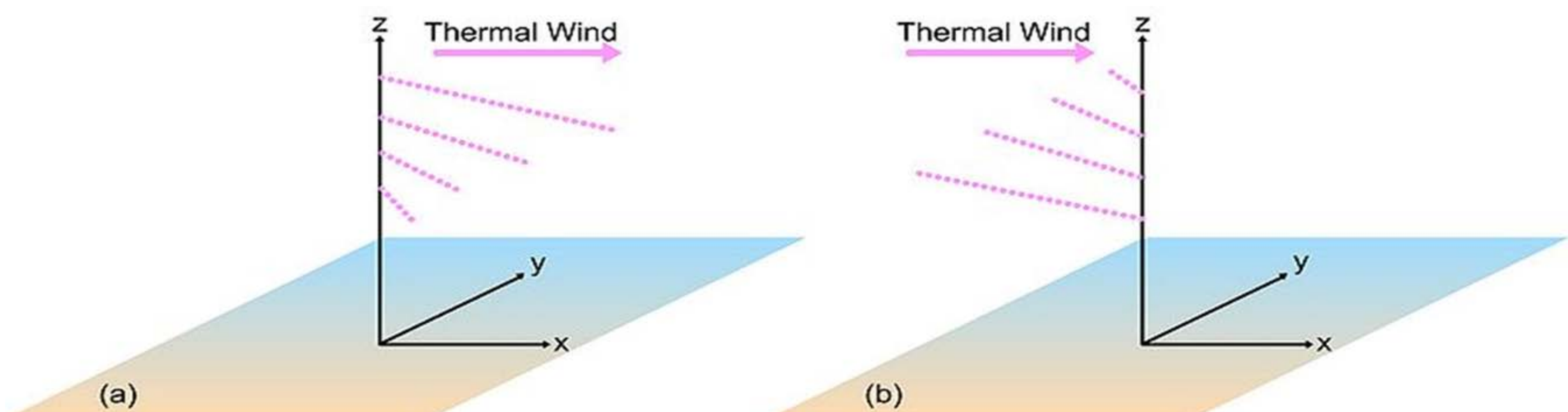
$$fv = \frac{R_d T}{p}\frac{\partial p}{\partial x} = R_d T \frac{\partial lnp}{\partial x}$$

$$-\frac{g}{R_d T} = \frac{\partial lnp}{\partial z}$$

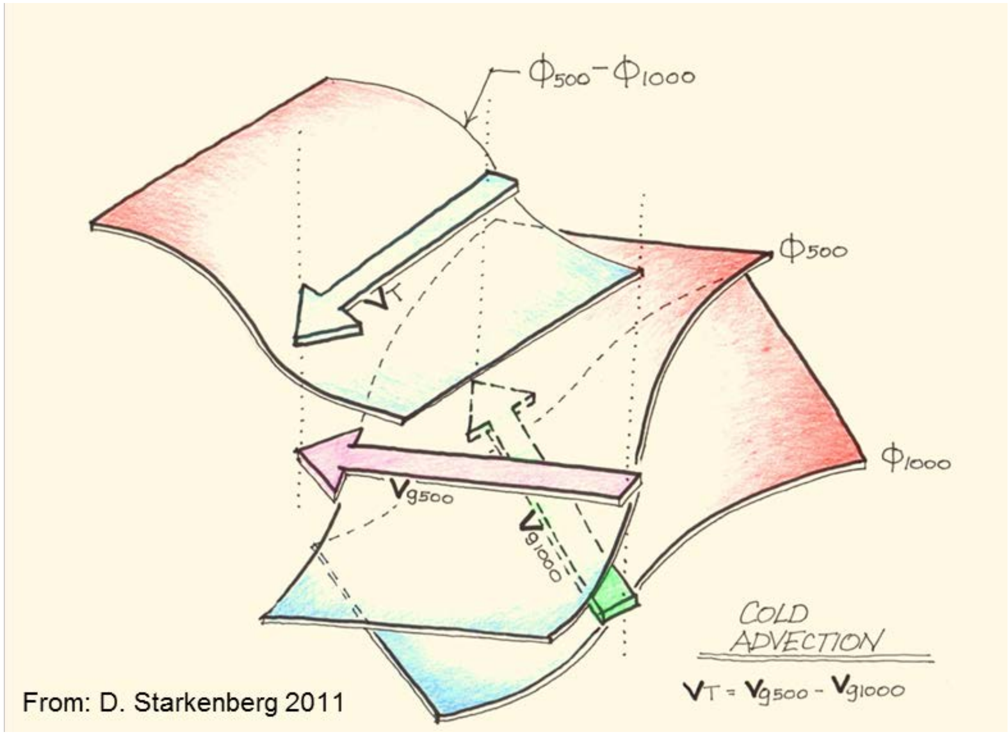
$$f\frac{\partial u}{\partial z} \approx -\frac{g}{T}\frac{\partial T}{\partial y}$$

$$f\frac{\partial v}{\partial z} \approx \frac{g}{T}\frac{\partial T}{\partial x}$$



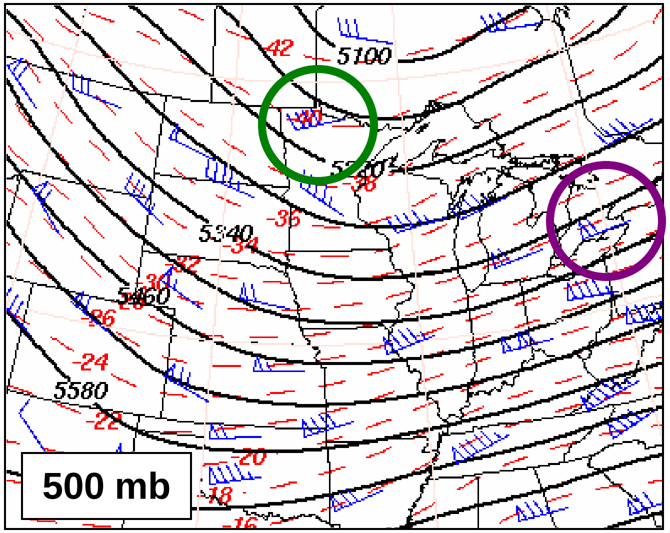


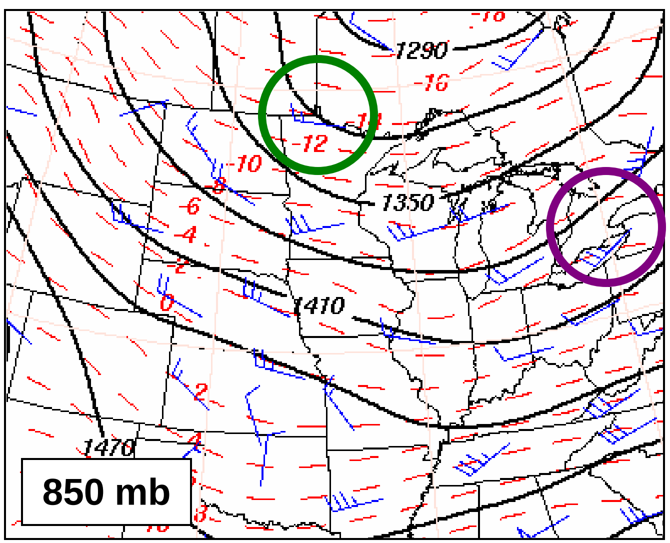
From: http://en.wikipedia.org/wiki/Thermal_wind



From: D. Starkenberg 2011

Thermal wind in synoptic application of pressure maps



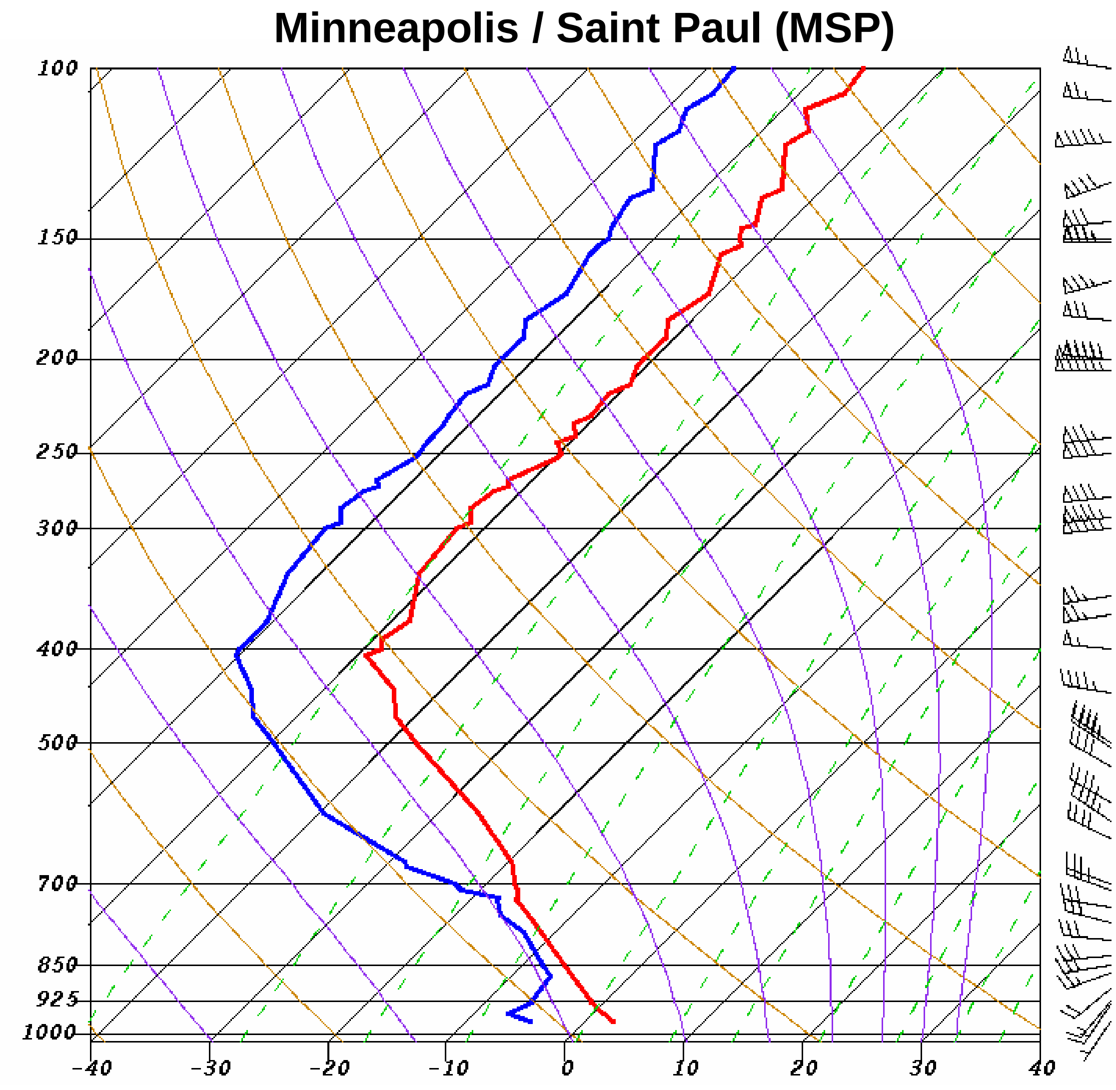


International Falls, MN

Buffalo, NY

Modified after Eastin

Thermal wind in synoptic application of soundings



Winds are backing with height → **Cold Air Advection**

Winds are veering with height → **Warm Air Advection**

Modified after: Eastin

Continuity equation

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}$$

mass divergence form

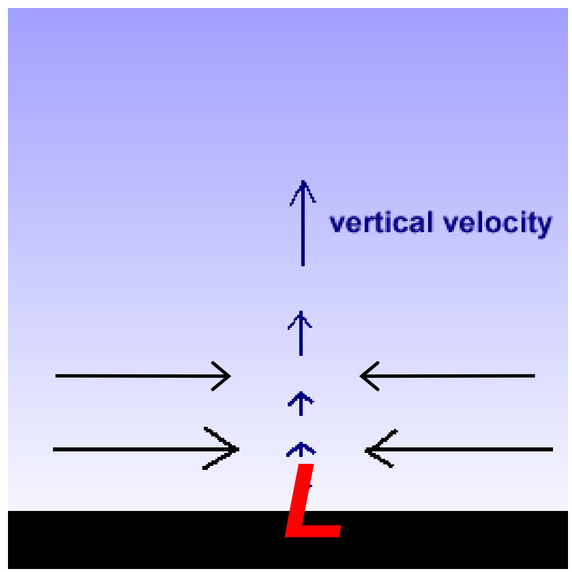
$$-\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

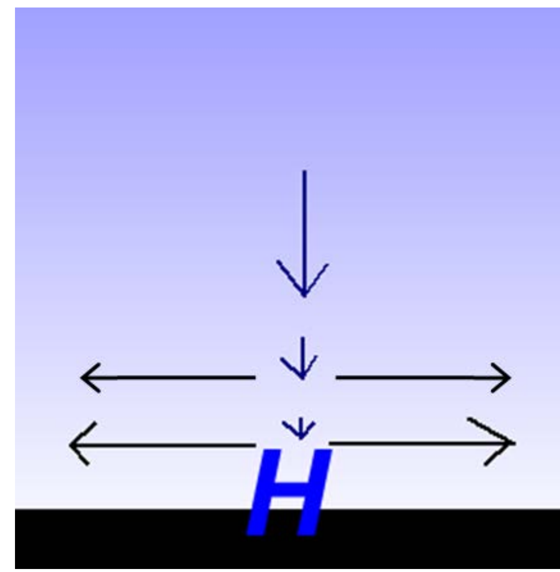
velocity divergence form

$$\boxed{0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}$$

$$\frac{\partial w}{\partial z} = -\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]$$

$$\frac{\partial w}{\partial z} = -\left[\nabla \bullet \mathbf{V}_h\right]$$





Primitive equations

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v} + \mathbf{J}) = -\nabla p - \rho\nabla\Phi - 2\rho\,\Omega\times\mathbf{v}$$

Momentum equation (Newton's 2nd law)

local change

Coriolis force

$$\frac{\partial(\rho\Theta)}{\partial t} + \nabla\cdot(\rho\mathbf{v}\Theta + \mathbf{J}_\Theta) = S_\Theta$$

Energy equation (1st law of thermodynamics; conservation of energy)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Continuity equation (conservation of mass)

advection

sources/sinks (phase transitions)

$$\frac{\partial(\rho q_n)}{\partial t} + \nabla\cdot(\rho\mathbf{v}q_n + \mathbf{J}_{q_n}) = S_{q_n} \qquad n = 0,\ldots,3$$

Balance equations for dry air and water substances

$$\frac{\partial(\rho\chi_m)}{\partial t} + \nabla\cdot(\rho\mathbf{v}\chi_m + \mathbf{J}_{\chi_m}) = S_{\chi_m} \qquad m = 1,\ldots,M$$

Balance equations for M species and particles

$$p = \rho R_0 T_v$$

sources/sinks (reactions, photolysis)

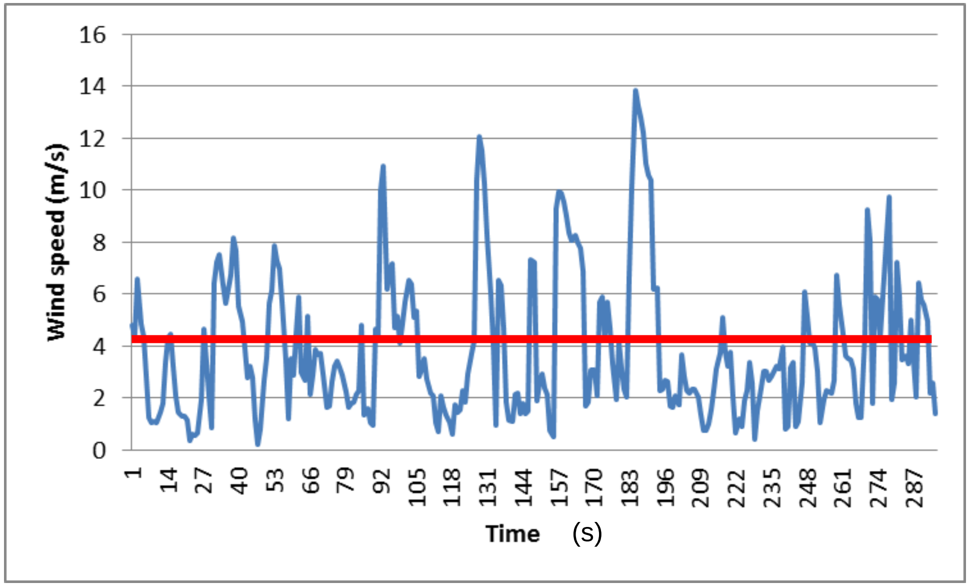
Equation of state

Mean and perturbation plus Reynolds-averaging

$$p = \overline{p} + p'$$

$$u = \overline{u} + u'$$

$$\rho = \overline{\rho} + \rho'$$



Mean and perturbation plus Reynolds-averaging

$$\overline{\varphi + \gamma} = \overline{\varphi} + \overline{\gamma}$$

$$\overline{\varphi\gamma} = \overline{\varphi}\overline{\gamma} + \overline{\varphi'\gamma'}$$

$$\overline{\alpha\varphi} = \alpha\overline{\varphi} \text{ with } \alpha = const.$$

$$\overline{\overline{\varphi}} = \overline{\varphi}$$

$$\overline{\frac{\partial\varphi}{\partial t}} = \frac{\partial\overline{\varphi}}{\partial t}$$

$$\overline{\nabla \cdot \varphi} = \nabla \cdot \overline{\varphi}$$

$$\overline{\nabla\varphi} = \nabla\overline{\varphi}$$

$$\overline{\varphi'} = 0$$

Example: Bousinesq approximation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + \mathbf{J}) = -\nabla p - \rho \nabla \Phi - 2\rho\, \Omega \times \mathbf{v}$$

$$\frac{\partial \bar\rho}{\partial t} + \nabla \cdot (\bar\rho\, \bar{\mathbf{v}} + \underbrace{\overline{\rho'\, \mathbf{v}'}}_{neglected}\ ) = \frac{\partial \bar\rho}{\partial t} + \nabla \cdot (\bar\rho\, \bar{\mathbf{v}}) = 0$$

$$\frac{\partial (\bar\rho\, \bar{\mathbf{v}} + \overbrace{\overline{\rho' \mathbf{v}'}}^{neglected}\ )}{\partial t} + \nabla \cdot \left(\bar\rho\, \bar{\mathbf{v}}\, \bar{\mathbf{v}} + \bar\rho\, \overline{\mathbf{v}'\mathbf{v}'} + \underbrace{\overline{\rho' \mathbf{v}'}\, \bar{\mathbf{v}} + \bar{\mathbf{v}}\, \overline{\rho' \mathbf{v}'} + \overline{\rho' \mathbf{v}' \mathbf{v}'}}_{neglected} + \bar{\mathbf{J}} \right)$$
$$= -\nabla \bar p - \bar\rho\, \nabla \phi - 2\bar\rho\, (\Omega \times \bar{\mathbf{v}}) - \underbrace{2\,\Omega \times \left(\overline{\rho' \mathbf{v}'}\right)}_{neglected}$$

identity tensor

Stokes friction tensor

Reynolds stress tensor

$$\frac{d\bar\rho}{dt} = -\,\bar\rho\, \nabla \cdot \bar{\mathbf{v}}$$

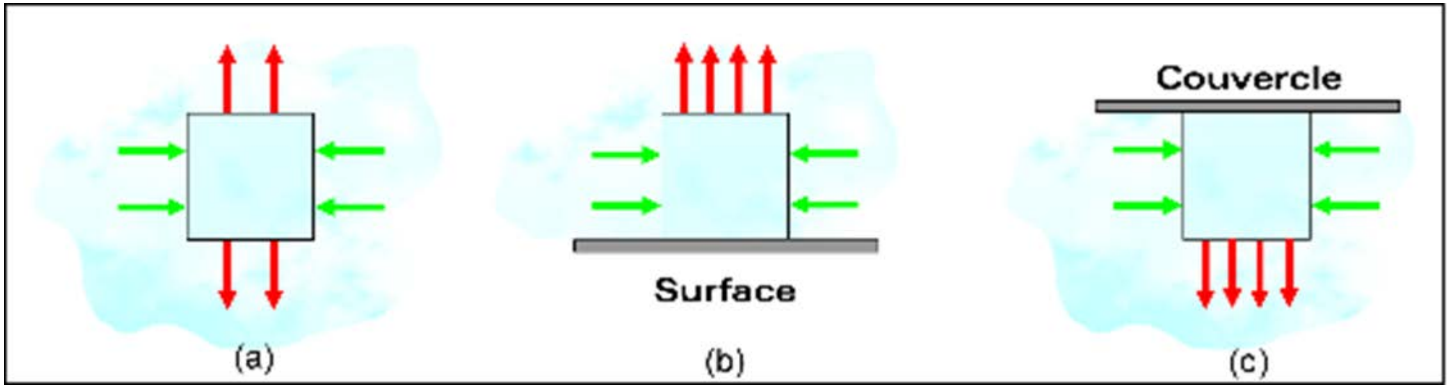
$$\bar\rho\, \frac{d\bar{\mathbf{v}}}{dt} + \nabla \cdot \left(\bar p\, \mathbf{E} + \bar{\mathbf{J}} + \mathbf{F} \right) = -\,\bar\rho\, \nabla \phi - \,2\,\bar\rho\, \Omega \times \bar{\mathbf{v}}$$

Anelastic approximation

$$\frac{d\rho}{dt} = -\rho\nabla\mathbf{u} \quad \Rightarrow \quad \nabla\left(\rho_{ref}\mathbf{u}\right) = 0$$

ρ fully prognostic

$\rightarrow \rho$ diagnostic (ideal gas law)



Closure problem – 1st order closure

$$\mathbf{F} = \bar{\rho}\,\overline{\mathbf{v}'\,\mathbf{v}'} = -\bar{\rho}\,\mathbf{K}_m : \left(\nabla\bar{\mathbf{v}} + (\nabla\bar{\mathbf{v}})^T\right)$$

$$\mathbf{H} = c_{p,0}\,\bar{\rho}\,\overline{\mathbf{v}'\,\Theta'} = -c_{p,0}\,\bar{\rho}\,\mathbf{K}_h \cdot \nabla\bar{\Theta}$$

$$\mathbf{F}_{q_n} = \bar{\rho}\,\overline{\mathbf{v}'\,q_n{}'} = -\,\bar{\rho}\,\mathbf{K}_{q_n} \cdot \nabla\overline{q_n} \quad for\, n = 1,2,3$$

$$\mathbf{F}_{\chi_m} = \bar{\rho}\,\overline{\mathbf{v}'\,\chi_m{}'} = -\bar{\rho}\,\mathbf{K}_{\chi_m} \cdot \nabla\overline{\chi_m} \quad form = 1,\ldots,M\ .$$

References

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